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Abstract Besov Space Approach to the Nonstationary Navier-Stokes Equations

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0. Introduction

The Navier-Stokes equations arising from viscous incompressible fluid dynamics has been investigated in depth. We consider the initial value problem of the Navier-Stokes equations

$$
\text{(I)} \quad
\begin{aligned}
u_t(x,t) + (u,\nabla)u(x,t) - \Delta u(x,t) &= f(x,t) - \nabla p(x,t) && \text{in } \Omega \times (0,T), \\
\nabla\cdot u(x,t) &= 0 && \text{in } \Omega \times (0,T), \\
u(x,t) &= 0 && \text{on } \Gamma \times (0,T), \\
u(x,0) &= a(x) && \text{in } \Omega.
\end{aligned}
$$

Here and hereafter $u = \{u_j(x,t)\}_{j=1}^n$ is the velocity field, $p = p(x,t)$ the pressure, $a = \{a_j(x)\}_{j=1}^n$ the initial velocity, $f = \{f_j(x,t)\}_{j=1}^n$ the external force, $u_t = \frac{\partial u}{\partial t}$, $\nabla = \{\frac{\partial}{\partial x_j}\}_{j=1}^n$, and Δ is the Laplacian. u and p are unknown, while f and a are given functions.

We always assume that Ω is a bounded domain in R^n with $n \geq 2$, a half space of R^n with $n \geq 2$, or an exterior domain in R^n with $n \geq 3$, and that the boundary Γ of Ω is smooth.

Fujita and Kato [4], [12] and Sobolevskii [21] established an approach to this Problem by means of fractional powers and semigroups of operators. Later, Giga and Miyakawa [9] developed a good L_r-theory which is a generalization of L_2-theory of Fujita and Kato. They did not assumed that the initial velocity is regular, which was assumed before in [4], [10], [23] etc.

However, we found that by making use of abstrct Besov spaces (see § 2 for their definition) in stead of fractional powers we obtain better results. The advantages of this approach are the following: (i) We can prove an estimate of semigroups in abstract Besov spaces (see Lemma 3.1), which is better than the well-known estimate:

$$\|A^{\alpha}T(t)x\| \le Ct^{-\alpha+\beta}\|A^{\beta}x\| \quad \text{for } x \in \mathcal{D}(A^{\beta}),\ t > 0,\ \alpha > \beta.$$

(ii) The nonlinear term $P_r(u,\nabla)u$ can be easily estimated (see Lemma 5.1). (iii) We need only know that the negative of the Stokes operator $-A_r$ generates an analytic semigroup on X_r, and we need not prove the existence of the bounded inverse of A_r which is proved only when Ω is a bounded domain, so that we can treat an exterior domain and a half space at the same time. (iv) We need only use the real interpolation theory, hence need not make use of the estimate $\|A_r^{it}\|_{\mathfrak{L}(X_r)} \le C_\varepsilon e^{\varepsilon|t|}$ for any $t \in \mathbb{R}$, which is hard to be proven (cf.[6], [7], [8]).

To eliminate the term ∇p we make use of P_r, a continuous operator from $\mathbb{L}_r(\Omega)$ to

$X_r :=$ the closure of the space $\{u \in (C_o^\infty(\Omega))^n;\ \nabla\cdot u = 0\}$ in $\mathbb{L}_r(\Omega)$

which is identical on X_r and $P_r\nabla p = 0$. (The existence of P_r is proved in [2], [5], [16].) The Stokes operator A_r is defined by $A_r = -P_r\Delta$ with domain $\mathcal{D}(A_r) = X_r \cap \{u \in \mathbb{W}_r^2(\Omega);\ u = 0 \text{ on } \Gamma\}$, then $-A_r$ generates an analytic semigroup $\{T(t);\ t \ge 0\}$ in X_r ([2], [3], [6], [7]). Here $\mathbb{W}_r^m(\Omega) = \{W_r^m(\Omega)\}^n$ is the Sobolev space and $\mathbb{L}_r(\Omega) = \{L_r(\Omega)\}^n$.

Applying P_r to (I), we get an abstract ordinary differential equation in X_r:

(II) $\qquad u_t + A_r u = F(u,u) + P_r f \quad t > 0,\ u(0) = a,$

where $F(u,v) = -P_r(u,\nabla)v$, whose integral form is the equation

(III) $\qquad u(t) = T(t)a + \int_0^t T(t-s)\{F(u(s),u(s)) + P_r f(s)\}ds, \quad t > 0.$

To solve (II) or (III), we extend $T(t)$ and $F(u,v)$ by continuity (see Lemma 3.1 and Lemma 5.1).

Our main results are the following:

Theorem A. If $a \in D^{\gamma}_{\infty-}(A_r)$, $P_r f(s) \in C_{1-\gamma-\delta}((0,T];D^{-\delta}_{\infty}(A_r))$, $\frac{n}{2r} - \frac{1}{2} \le \gamma < 1$, $0 < \gamma+\delta < 1$, and $\delta < 1$, then there exist a positive number T_0 and a non-negative number $\alpha > \gamma$ such that there is a unique solution $u \in C([0,T_0];D^{\gamma}_{\infty-}(A_r)) \cap C_{\alpha-\gamma}((0,T_0];D^{\alpha}_1(A_r))$ of (III).

Any solution u of (III) satisfying

$u \in C([0,T_0];D^{\gamma}_{\infty-}(A_r)) \cap C_{\sigma-\gamma}((0,T_0],D^{\sigma}_{\infty}(A_r))$ for some $\sigma > \gamma^+$

is unique. Here $\gamma^+ = \max\{\gamma,0\}$, $D^{\alpha}_q(A)$ denotes the abstract Besov space defined in § 2, $C(I;Y)$ denotes the space of Y-valued continuous functions on an interval I, and

$C_{\gamma}((0,T];Y) := \{u \in C((0,T];Y);\ \|u(t)\|_Y = o(t^{-\gamma}) \text{ as } t \to 0\}.$

Theorem B. Under the assumptions of Theorem A, let u be a solution of (III) belonging to $C((0,T];D^{\sigma}_1(A_r))$ for some non-negative number σ with $\sigma > \gamma$. Then

(i) $\qquad u \in C^{1-\alpha-\delta}((0,T];D^{\alpha}_1(A_r))$ for any $0 \le \alpha < 1 - \delta$.

(ii) Furthermore, if $P_r f \in C^{\nu}((0,T];X_r)$, $\nu > 0$, then u is a solution of (II), namely, u(t) is differentiable in $0 < t < T$, $u(t) \in \mathcal{D}(A_r)$ for $0 < t < T$ and satisfies (II).

Here $C^{\mu}(I;Y)$ denotes the space of Y-valued (locally) μ-Hölder continuous functions on I.

Theorem C. Under the assumptions of Theorem A, assume that $P_r f \in$

$\{C^\infty(\bar{\Omega}\times(0,T])\}^n$. Then, any solution u of (Ⅲ) in $C((0,T];D_1^\sigma(A_r))$ for some non-negative number σ with $\sigma > \gamma$ belongs to $\{C^\infty(\bar{\Omega}\times(0,T])\}^n$, where $C^\infty(\Omega)$ denotes the space of infinitely differentiable functions on an open set Ω.

These results are improvements of those in Fujita and Kato [4], and in Giga and Miyakawa [9]. For instance,

Result in Fujita and Kato [4]. Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$ and let $1/4 < \gamma < 1/2$. Assume that $a \in \mathcal{D}(A_2^\gamma)$ and that $\|P_r f(t)\|_2 = o(t^{-1+\gamma})$ as $t \to 0$. Then there exists a unique solution u of (Ⅲ) such that (i) $u \in C([0,T_*];X_2)$, (ii) $u \in C((0,T_*];\mathcal{D}(A_2^\alpha))$ for any $3/4 < \alpha < \gamma + 1/2$, and that (iii) $\|A_2^\alpha u(t)\|_2 = O(t^{\gamma-\alpha})$ as $t \to 0$, where we simply denote the norm of $\mathbb{L}_r(\Omega)$ by $\|\cdot\|_r$. Here T_* is a positive number depending on γ, α, $\|A_2^\gamma a\|_2$ and $\sup_{0<s\leq T} s^{1-\gamma}\|P_2 f(s)\|_2$.

Result in Giga and Miyakawa [9]. Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$, and let $n/2r - 1/2 \leq \gamma < 1$, $-\gamma < \delta < 1 - |\gamma|$ and $\delta \geq 0$. Assume that $a \in \mathcal{D}(A_r^\gamma)$ and $\|A_r^{-\delta}P_r f(t)\|_r$ is continuous on $(0,T)$ and satisfies $\|A_r^{-\delta}P_r f(t)\|_r = o(t^{\gamma+\delta-1})$ as $t \to 0$. Then for any $\gamma < \alpha < 1-\delta$ there is a solution $u \in C([0,T_*];\mathcal{D}(A_r^\gamma))\cap C_{\alpha-\gamma}((0,T_*];\mathcal{D}(A_r^\alpha))$ of (Ⅲ). Here T_* depends on γ, δ, α, a and $P_r f$.

The conditions required to the initial velocity and the external force in Theorem A are weaker than those in [9] and more precise information about solutions are contained in this theorem.

Notations. We will use the following notations: For an open set Ω in $\mathbb{R}^n$ and $1 \leq p < \infty$ we define

$$\|f\|_{L_p(\Omega)} := \{\int_\Omega |f(x)|^p dx\}^{1/p}, \quad \|f\|_{L_p^*(\Omega)} := \{\int_\Omega |f(x)|^p |x|^{-n} dx\}^{1/p},$$

and for $p = \infty$ make the usual modification. $L_p(\Omega)$ (or $L_p^*(\Omega)$) denotes the space of all measurable functions f with $\|f\|_{L_p(\Omega)} < \infty$ (or $\|f\|_{L_p^*(\Omega)} < \infty$). For a Banach space X we denote by $L_p(\Omega;X)$ (or $L_p^*(\Omega;X)$) the set of all strongly measurable X-valued functions with $\|f(x)\|_X \in L_p(\Omega)$ (or $L_p^*(\Omega)$). We also consider the spaces with the exponent $\infty-$. Namely,

$$L_{\infty-}(\Omega;X) \ (= L_{\infty-}^*(\Omega;X) \ ) := \{ \ f \in L_\infty(\Omega;X); \ \|f(x)\|_X \to 0 \text{ as } |x| \to \infty \ \},$$

and its norm is that of L_∞. We define $p < \infty- < \infty$ for real number p.

$$W_p^m(\Omega) := \{f \in L_p(\Omega); \ \partial^\alpha f \in L_p(\Omega) \text{ for any multi-index with } |\alpha| \le m\},$$

where $\partial^\alpha f$ denotes the weak derivative of f, $|\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_n$, and its norm is given by $\|f\|_{W_p^m(\Omega)} := \sum_{|\alpha|\le m} \|\partial^\alpha f\|_{L_p(\Omega)}$.

$\mathfrak{L}(X,Y)$ denotes the space of all continuous linear operators from X to Y, $\mathfrak{L}(X) := \mathfrak{L}(X,X)$, and $\mathfrak{D}(A)$ denotes the domain of an operator A.

1. Besov spaces

Here we describe the definition and some properties of Besov spaces, which are one of our main tools.

Definition 1. Let Ω be an open set in $\mathbf{R}^n$, and let $1 \le p, \ q \le \infty$. When $0 < \sigma \le 1$ we define

$$B_{p,q}^{\sigma}(\Omega) := \{f \in L_p(\Omega); \ |f|_{B_{p,q}^{\sigma}(\Omega)} < \infty \ \}, \tag{1.1}$$

$$|f|_{B_{p,q}^{\sigma}(\Omega)} := \|\{|y|^{-\sigma}\|f(\cdot+y)-f(\cdot)\|_{L_p(\Omega\cap(\Omega-y))}\}\|_{L_q^*(\mathbf{R}^n)} \quad \text{if } \sigma < 1, \tag{1.2}$$

$$|f|_{B_{p,q}^{1}(\Omega)} := \|\{|y|^{-1}\|f(\cdot+2y)-2f(\cdot+y)+f(\cdot)\|_{L_p(\Omega_{2,y})}\}\|_{L_q^*(\mathbf{R}^n)},$$

where $\Omega_{2,y} = \Omega\cap(\Omega-y)\cap(\Omega-2y)$, and its norm is given by

$$\|f\|_{B_{p,q}^{\sigma}(\Omega)} := |f|_{B_{p,q}^{\sigma}(\Omega)} + \|f\|_{L_p(\Omega)}. \tag{1.3}$$

When $\sigma > 1$, by expressing $\sigma = k + \theta$, $k \in \mathbb{N}$, $0 < \theta \le 1$, we define

$$(1.4)\quad B_{p,q}^{\sigma}(\Omega) := \{f \in W_p^k(\Omega);\ \partial^\alpha f \in B_{p,q}^{\theta}(\Omega) \text{ for every } |\alpha| = k\},$$

$$(1.5)\quad |f|_{B_{p,q}^{\sigma}(\Omega)} := \sum_{|\alpha|=k} |\partial^\alpha f|_{B_{p,q}^{\theta}(\Omega)},$$

$$(1.6)\quad \|f\|_{B_{p,q}^{\sigma}(\Omega)} := |f|_{B_{p,q}^{\sigma}(\Omega)} + \|f\|_{W_p^m(\Omega)}.$$

It is easy to see that $B_{p,q}^{\sigma}(\Omega)$ are all Banach spaces.

Lemma 1. Let $1 \le p,\ q \le \infty$, $1 \le \xi,\ \eta \le \infty$, $\lambda = n/p - n/q$, $\sigma \in \mathbf{R}$, $\tau \in \mathbf{R}$, and let Ω be an open set with the cone property.

(i) (Imbedding). If $p \le q$ and if $\tau > \sigma+\lambda$, then $B_{p,\xi}^{\tau}(\Omega) \subset B_{q,\eta}^{\sigma}(\Omega)$, $B_{p,\xi}^{\tau}(\Omega) \subset W_q^{\sigma}(\Omega)$, $W_p^{\tau}(\Omega) \subset B_{q,\eta}^{\sigma}(\Omega)$, $W_p^{\tau}(\Omega) \subset W_q^{\sigma}(\Omega)$. We also have

$$(1.7)\quad B_{p,\xi}^{\sigma+\lambda}(\Omega) \subset B_{q,\eta}^{\sigma}(\Omega) \quad \text{if } \xi \le \eta,$$

$$(1.8)\quad B_{p,\xi}^{\sigma+\lambda}(\Omega) \subset W_q^{\sigma}(\Omega) \quad \text{if } \xi \le q < \infty \text{ or } \xi = 1,$$

$$(1.9)\quad W_p^{\sigma+\lambda}(\Omega) \subset B_{q,\eta}^{\sigma}(\Omega) \quad \text{if } 1 < p < q,\ p \le \eta,$$

$$(1.10)\quad W_p^{\sigma+\lambda}(\Omega) \subset W_q^{\sigma}(\Omega) \quad \text{if } 1 < p < q < \infty.$$

(ii) (Real Interporation). Let $0 < \theta < 1$, $\mu = (1-\theta)\sigma + \theta\tau$. Then

$$(1.11)\quad (B_{p,\xi}^{\sigma}(\Omega), B_{p,\eta}^{\tau}(\Omega))_{\theta,q} = (W_p^{\sigma}(\Omega), W_p^{\tau}(\Omega))_{\theta,q} = B_{p,q}^{\mu}(\Omega).$$

Here $(\ ,\ )_{\theta,q}$ denotes the real interpolation space.

(iii) (Product in Besov Spaces). Let $\gamma,\ \sigma,\ \tau > 0$ and assume that $\gamma \le \min\{\sigma,\tau,\sigma+\tau-n/r\}$. Then, for any $u \in B_{r,q}^{\sigma}(\Omega)$ and $v \in B_{r,q}^{\tau}(\Omega)$ we have

$$(1.12)\quad \|uv\|_{B_{r,q}^{\gamma}} \le C\|u\|_{B_{r,q}^{\sigma}} \cdot \|v\|_{B_{r,q}^{\tau}}.$$

Proof. cf. Muramatu [17],[18],[19].

2. Abstract Besov Spaces

Abstract Besov spaces have been introduced and precisely investigated by Komatsu [13],[14],[15] for a non-negative operator A in a Banach space X. Our definition of the space $D_p^{\sigma}(A)$ is slightly different from that of Komatsu, which make it possible to treat systematically all the spaces $D_p^{\sigma}(A)$, $-\infty < \sigma < \infty$.

Throughout this section and the next section by $\|x\|$ and $\|T\|$ we denote the norm of X and $\mathfrak{L}(X)$, respectively.

Definition 2. A closed linear operator A in X is called **non-negative** if there is a number $c_0 \geq 0$ such that $(-\infty, -c_0)$ is contained in the resolvent set of A and if

(2.1) $$M := \sup\{\|\lambda(\lambda+A)^{-1}\|;\ \lambda > c_0\} < \infty.$$

For simplicity we assume always that $c_0 < 1$ in this paper.

For a non-negative operator A, real number σ and $1 \leq p \leq \infty$ (including $p = \infty-$) we define the space $D_p^\sigma(A)$ by the completion of the space $\{x \in X;\ \lambda^\sigma \lambda^\ell A^n(\lambda+A)^{-\ell-n}x \in L_p^*([1,\infty);X)\}$ with respect to the norm $\|\cdot\|_{D_p^\sigma(A)}$, where n and ℓ are the least non-negative integers such that $n > \sigma > -\ell$, and

(2.2) $$\|x\|_{D_p^\sigma(A)} := |x|_{D_p^\sigma(A)} + \|(1+A)^{-\ell}x\|,$$

(2.3) $$|x|_{D_p^\sigma(A)} := \|\lambda^\sigma \lambda^\ell A^n(\lambda+A)^{-\ell-n}x\|_{L_p^*([1,\infty);X)}.$$

For the case $p = \infty$, $\sigma \leq 0$ we have to make some modifications.

Lemma 2.1. Let A be a non-negative operator in X and let k and m be positive integers. Then for any x in $\overline{\mathfrak{D}(A)}$ and $\kappa \geq 1$ we have

(2.4) $$x = c_{m,k}\int_\kappa^\infty \lambda^{k-1}A^m(\lambda+A)^{-k-m}x\,d\lambda + Q_{m,k}(A(\kappa+A)^{-1})\kappa^k(\kappa+A)^{-k}x,$$

where $Q_{m,k}(t) = \sum_{j=0}^{m-1}\binom{k+j-1}{j}t^j$, and $c_{m,k} = m\binom{m+k-1}{m}$.

Proof. This follows from the identity

(2.5) $$\frac{d}{d\mu}\{Q_{m,k}(A(\mu+A)^{-1})\mu^k(\mu+A)^{-k}\} = c_{m,k}\mu^{k-1}A^m(\mu+A)^{-k-m}$$

and the mean ergodic theorem (cf. K.Yosida [24] p.217).

Using this lemma, arguments analogous to those in Komatsu [13], [14],[15], (see also [20]). yield the following

Lemma 2.2. (Basic Properties of Abstract Besov Spaces). Let σ be a real number, m and k integers, and let $1 \le p \le \infty$.

(i) Assume that k and m are non-negative and $-k < \sigma < m$. Then $x \in X$ belongs to $D_p^\sigma(A)$ if and only if $\lambda^\sigma \lambda^k A^m(\lambda+A)^{-k-m}x \in L_p^*([1,\infty);X)$, and the norm of $D_p^\sigma(A)$ is equivalent to the norm

$$\|\lambda^\sigma \lambda^k A^m(\lambda+A)^{-k-m}x\|_{L_p^*([1,\infty);X)} + \|(1+A)^{-k}x\|. \tag{2.6}$$

In particular, if $0 < \sigma < m$, then

$$D_p^\sigma(A) = \{x \in X;\ \lambda^\sigma A^m(\lambda+A)^{-m}x \in L_p^*([1,\infty);X)\},$$

and its norm is equivalent with

$$\|\lambda^\sigma A^m(\lambda+A)^{-m}x\|_{L_p^*([1,\infty);X)} + \|x\|, \tag{2.7}$$

while $D_p^{-\sigma}(A)$, $1 \le p \le \infty-$, is the completion of X with respect to the norm

$$\|\lambda^{-\sigma}\lambda^m(\lambda+A)^{-m}x\|_{L_p^*([1,\infty);X)} + \|(1+A)^{-m}x\|, \tag{2.8}$$

and for any $x \in X$ its norm in $D_p^{-\sigma}(A)$ is equivalent with this norm.

(ii) If $\sigma > \tau$ or if $\sigma = \tau$ and $p \le q \le \infty$, then

$$D_p^\sigma(A) \subset D_q^\tau(A) \quad \text{with continuous inclusion.} \tag{2.9}$$

(iii) Set $D^0(A) = X$ and for a positive integer n $D^n(A) = \mathcal{D}(A^n)$ with norm $\|x\|_{D^n(A)} = \|A^n x\| + \|x\|$, and define $D^{-n}(A)$ by the completion of X with respect to the norm $\|(1+A)^{-n}x\|$. Then

$$D_1^m(A) \subset D^m(A) \subset D_\infty^m(A) \quad \text{with continuous inclusions, and if} \tag{2.10}$$

$\mathcal{D}(A)$ is dense in X $D^m(A) \subset D_{\infty-}^m(A)$.

(iv) If $\sigma < m$, $m > 0$ and $p \le \infty-$, then $\mathcal{D}(A^m)$ is dense in $D_p^\sigma(A)$.

(v) If $0 < \theta < 1$ and $k \ne m$, then

$$D_p^{k(1-\theta)+m\theta}(A) = (D^k(A), D^m(A))_{\theta,p}. \tag{2.11}$$

Remark 2. For a positive number σ the space $D_p^\sigma(A)$ coincides with that defined by Komatsu [14], and the norm (2.8) is apparently similar to that of the space $R_p^{\sigma}(A)$ introduced by Komatsu [15], but the space $D_p^{-\sigma}(A)$ is different from $R_p^{\sigma}(A)$.

3. Semigroups and abstract Besov spaces

In this section we always assume that $-A$ generates an **analytic semigroup** $\{T(t);t\geq 0\}$ in X, and estimate the norm of $T(t)$ as an operator acting between abstract Besov spaces relative to A. As stated in Definition 2, $A^mT(t)$, $t > 0$, $m = 0,1,\cdots$, can be extended to a unique linear operator on $\cup_{n=0}^{\infty}D^{-n}(A)$ which is bounded on $D^{-k}(A)$ for any k.

Lemma 3.1. If m is a positive integer and if $m + \alpha > \beta$, then $A^mT(t)$ maps $D_{\infty}^{\beta}(A)$ into $D_1^{\alpha}(A)$ and

(3.1) $$\|A^mT(t)x\|_{D_1^{\alpha}} \leq Ct^{\beta-m-\alpha}\|x\|_{D_{\infty}^{\beta}} \qquad \text{for } 0 < t \leq T < \infty.$$

Assume moreover that $x \in D_{\infty-}^{\beta}(A)$, then $\|A^mT(t)x\|_{D_1^{\alpha}} = o(t^{\beta-m-\alpha})$ as $t \to +0$, and $T(t)x \in C([0,T];D_{\infty-}^{\beta}(A))$.

Definition 3. For a real number γ, $\sigma = m+\theta \geq 0$, m an integer, $0 \leq \theta < 1$, and a Banach space Y the space $C_{\gamma}^{\sigma}((0,T];Y)$ is the space of all functions $g \in C^m((0,T];Y)$ such that

(3.5) $$|g|_{j,\gamma;Y,T} := \sup_{0<t\leq T} t^{j+\gamma}\|g^{(j)}(t)\|_Y, \quad j = 0,1,\cdots,m,$$

(3.6) $$|g|_{\sigma,\gamma;Y,T} := \sup_{h>0}\ \sup_{0\leq t\leq T-h} t^{\sigma+\gamma}h^{-\theta}\|g^{(m)}(t+h) - g^{(m)}(t)\|_Y,$$

are finte, and its norm is defined by

(3.7) $$\|g\|_{\sigma,\gamma;Y,T} := \textstyle\sum_{j=0}^{m}|g|_{j,\gamma;Y,T} + |g|_{\sigma,\gamma;Y,T},$$

where $g^{(j)}$ denotes the j-th derivative of g.

Lemma 3.2. Let $T_0 > 0$, Y and Z Banach spaces, and assume that $Z \subset Y \subset D^{-n}(A)$ for some n with continuous inclusions and that

(3.8) $\|A^m T(t)\|_{\mathfrak{L}(Y,Z)} \le Ct^{-m-\kappa}$ for any $0 < t \le T_0$ and $m = 0,1,2,\cdots$, where C and $0 \le \kappa < 1$ are constants. Let $g \in C_\gamma^\sigma((0,T];Y)$, $\sigma \ge 0$, $0 \le \gamma < 1$, $0 < T \le T_0$ and assume that $\sigma-\kappa$ is fractional. Then

(3.9) $$v(t) = \int_0^t T(t-s)g(s)ds.$$

belongs to $C_{\gamma+\kappa-1}^{\sigma-\kappa+1}((0,T];Z)$ and

(3.10) $$\|v\|_{\sigma-\kappa+1,\gamma+\kappa-1,Z,T} \le C\|g\|_{\sigma,\gamma,Y,T},$$

where C is a positive constant independent of g and T.

In particular, if $g \in C_\gamma^\sigma((0,T];D_\infty^\beta(A))$, $\beta < \alpha < \beta+1$, then $v \in C_{\gamma+\alpha-\beta-1}^{\sigma+\beta-\alpha+1}((0,T];D_1^\alpha(A))$.

4. The basic properties of the Stokes operator

In this section we always assume that $\alpha > 0$, $1 < r < \infty$ and $1 \le q \le \infty$, and A_r denotes the Stokes operator, and $\mathbb{B}_{r,q}^\alpha(\Omega) = \{B_{r,q}^\alpha(\Omega)\}^n$.

Lemma 4.1. $P_r \in \mathfrak{L}(\mathbb{B}_{r,q}^\alpha(\Omega))$.

Lemma 4.2. If $u \in \mathcal{D}(A_r)$ and $A_r u \in \mathbb{B}_{r,q}^\alpha(\Omega)$, then $u \in \mathbb{B}_{r,q}^{\alpha+2}(\Omega)$ and

(4.1) $$\|u\|_{\mathbb{B}_{r,q}^{\alpha+2}(\Omega)} \le C\{\|A_r u\|_{\mathbb{B}_{r,q}^\alpha(\Omega)} + \|u\|_{\mathbb{L}_r(\Omega)}\}.$$

Lemma 4.3. We have

(4.2) $$D_q^\alpha(A_r) \subset X_r \cap \mathbb{B}_{r,q}^{2\alpha}(\Omega),$$

and for any poisitive integer k and for any $\lambda \ge 1$

(4.3) $$\|\lambda^k(\lambda+A_r)^{-k}\|_{\mathfrak{L}(X_r, D_q^\alpha(A_r))} \le C\lambda^\alpha.$$

Lemma 4.4. For any $1 \le \lambda$ we have

(4.4) $$\|\partial_j(\lambda+A_r)^{-1}\|_{\mathfrak{L}(X_r,\mathbb{L}_r(\Omega))} \le C\lambda^{-1/2},$$

(4.5) $$\|(\lambda+A_r)^{-1}P_r\partial_j\|_{\mathfrak{L}(\mathbb{L}_r(\Omega),X_r)} \le C\lambda^{-1/2}.$$

Lemma 4.5. Let $1 < s < r \le \infty$, $2k \ge 2\rho \ge \frac{n}{s} - \frac{n}{r}$ and $k \in N$. Then

(4.6) $$\|\lambda^k(\lambda+A_s)^{-k}\|_{\mathfrak{L}(X_s,\mathbb{L}_r(\Omega))} \le C\lambda^\rho \qquad \text{for } 1 \le \lambda < \infty.$$

Lemma 4.6. Let $1 < s < r < \infty$, $2\rho \ge \frac{n}{s} - \frac{n}{r}$ and $\beta \in \mathbf{R}$. Then

(4.7) $$D_q^{\beta}(A_s) \subset D_q^{\beta-\rho}(A_r).$$

5. Estimation of the nonlinear term

The inequality for the nonlinear term $P_r(u,\nabla)u$ by means of abstract Besov spaces, which is proved in the following, is a crucial result in our investigation. Giga and Miyakawa [9] have given a similar estimate by means of fractional powers A_r^{α} $(\alpha>0)$ and $A_r^{-\delta}$ $(\delta>0)$, but their estimate holds only when $\delta+\rho > 1/2$ and $\delta < 1/2 + n/2 - n/2r$.

Lemma 5.1. Let δ, θ and ρ be numbers satisfying

(5.1) $$\theta+\rho+\delta \ge \frac{n}{2r} + \frac{1}{2},\ \theta+\rho > \frac{n}{r} - \frac{n}{2},\ \rho+\delta \ge \frac{1}{2},\ \delta \ge 0,\ \gamma \ge 0,\ \rho \ge 0.$$

Then, for any $u \in D_1^{\theta}(A_r)\cap\mathscr{D}(A_r)$ and $v \in D_1^{\rho}(A_r)\cap\mathscr{D}(A_r)$ we have

(5.2) $$\|P_r(u,\nabla)v\|_{D_\infty^{-\delta}(A_r)} \le C\|u\|_{D_1^{\theta}(A_r)}\cdot\|v\|_{D_1^{\rho}(A_r)}.$$

We can replace $D_\infty^{0}(A_r)$ by X_r when $\delta = 0$.

Since $\mathscr{D}(A_r^m)$ is dense in $D_1^{\theta}(A_r)$ and $D_1^{\rho}(A_r)$, by this lemma we can uniquely extend $P_r(u,\nabla)v$ to a continuous bilinear operator from $D_1^{\theta}(A_r)\times D_1^{\rho}(A_r)$ to $D_\infty^{-\delta}(A_r)$ if $\{\theta,\rho,\delta\}$ satisfies (5.1), and we denote its extension by $F_{\theta,\rho,\delta}(u,v)$. But, when $\{\theta',\rho',\delta'\}$ is another triple satisfying (5.1), $F_{\theta,\rho,\delta}(u,v) = F_{\theta',\rho',\delta'}(u,v)$ holds for any $(u,v) \in \mathscr{D}(A_r^m)\times\mathscr{D}(A_r^m)$, and for sufficiently large m $\mathscr{D}(A_r^m)\times\mathscr{D}(A_r^m)$ is dense in $D_1^{\theta}(A_r)\times D_1^{\rho}(A_r)$ and in $D_1^{\theta'}(A_r)\times D_1^{\rho'}(A_r)$, so it follows that

$$F_{\theta,\rho,\delta}(u,v) = F_{\theta',\rho',\delta'}(u,v)$$

holds for any $(u,v) \in \{D_1^{\theta}(A_r)\times D_1^{\rho}(A_r)\}\cap\{D_1^{\theta'}(A_r)\times D_1^{\rho'}(A_r)\}$. Namely, $F_{\theta,\rho,\delta}(u,v)$ is independent of the choice of $\{\rho,\theta,\delta\}$. Hence we omit these suffixes and write it simply as $F(u,v)$ in the following.

Lemma 5.2. Assume that γ, δ and ρ satisfy (5.1). If $u \in$

$C^{\mu}_{\eta}((0,T];\ D^{\theta}_{1}(A_r))$ and if $v \in C^{\mu}_{\eta}((0,T];D^{\rho}_{1}(A_r))$ with $\mu \geq 0$ and $\eta \geq 0$, then $F(u,v) \in C^{\mu}_{2\eta}((0,T];D^{-\delta}_{\infty}(A_r))$.

6. Proof of Theorem A

Now we are in a position to prove Theorem A. First note that it follows from the assumptions, Lemma 3.1 and Lemma 3.2 that

$$(6.1) \qquad u_0(t) := T(t)a + \int_0^t T(t-s)P_r f(s)ds$$

belongs to $C([0,T];D^{\gamma}_{\infty-}(A_r))\cap C_{\alpha-\gamma}((0,T];D^{\alpha}_{1}(A_r))$ for any α with $\gamma < \alpha$, $0 \leq \alpha < 1 - \delta$. We choose a number α so that

$$(6.2) \qquad \gamma < \alpha < 1-\delta,\ \alpha-\gamma < \frac{1}{2},\ \alpha < \frac{1}{2} + \frac{\gamma}{2},\ \alpha \geq 0,$$

and take a number β so that

$$(6.3) \qquad 1+\gamma \geq 2\alpha+\beta \geq \frac{n}{2r} + \frac{1}{2},\ 1 > \alpha+\beta \geq \frac{1}{2},\ \beta \geq 0.$$

Then, $2\alpha > 2\gamma \geq \frac{n}{r} - 1 \geq \frac{n}{r} - \frac{n}{2}$. Define Φv by

$$(6.2) \qquad \Phi v(t) = \int_0^t T(t-s)F(u_0(s)+v(s),u_0(s)+v(s))ds,$$

set $u = u_0 + v$ and substitute this into (III). Then it becomes $v = \Phi v$. Thus, a fixed point of Φ gives a solution of (III).

It follows from Lemma 5.2 that if $v \in C_{\alpha-\gamma}((0,T];D^{\alpha}_{1}(A_r))$ then $F(u_0+v,u_0+v) \in C_{2\alpha-2\gamma}((0,T];D^{\beta}_{\infty-}(A_r))$ and

$$(6.3) \qquad \|F(u_0+v,u_0+v)\|_{-\beta,\infty,2(\alpha-\gamma),t} \leq C_1\|u_0+v\|^2_{\alpha,1,\alpha-\gamma,t},$$

where $\|u\|_{\alpha,q,\gamma,t} := \|u\|_{C^0_{\gamma}((0,t];D^{\alpha}_{q}(A_r))}$ (see Definition 3). This means, with the aid of Lemma 3.2, that $\Phi v \in C_{\alpha-\gamma}((0,T];D^{\alpha}_{1}(A_r))$ and

$$(6.4) \quad t^{\alpha-\gamma-\eta}\|\Phi v(t)\|_{D^{\alpha}_{1}} \leq C_2\|F(u_0+v,u_0+v)\|_{-\beta,\infty,2\alpha-2\gamma,t}$$
$$\leq C_1C_2\{\|u_0\|_{\alpha,1,\alpha-\gamma,t} + \|v\|_{\alpha,1,\alpha-\gamma,t}\}^2,$$

where $\eta = 1+\gamma-2\alpha-\beta$.

When $\gamma > \frac{n}{2r} - \frac{1}{2}$, we can choose α and β so that $\eta > 0$, so we can take a number $T_0 \leq T$ so small that $4T_0^{\eta}C_1C_2\|u_0\|_{\alpha,1,\alpha-\gamma,T} < 1$. When $\gamma =$

$\frac{n}{2r} - \frac{1}{2}$, η must be 0. But, since Lemma 3.1 and Lemma 3.2 imply that $\|u_0\|_{\alpha,1,\alpha-\gamma,t} \to 0$ as $t \to +0$, there is $T_0 \in (0,T]$ such that $4C_1C_2\|u_0\|_{\alpha,1,\alpha-\gamma,T_0} < 1$.

Therefore, if $\|v\|_{\alpha,1,\alpha-\gamma,T_0} \le K_0 := \|u_0\|_{\alpha,1,\alpha-\gamma,T_0}$, then we have

$$(6.5)\qquad \|\Phi v\|_{\alpha,1,\alpha-\gamma,T_0} \le C_1C_2T_0^{\eta}(K_0 + K_0)^2 \le K_0.$$

Thus, Φ maps the space

$$M := \{v \in C_{\alpha-\gamma}((0,T_0];D_1^{\alpha}(A_r));\|v\|_{\alpha,1,\alpha-\gamma,T_0} \le K_0\}$$

into itself. Obviously M is a complete metric space. Also, we have by Lemma 5.1

$$(6.6)\qquad \|F(v_1(s),v_1(s)) - F(v_2(s),v_2(s))\|_{D_\infty^{-\beta}}$$
$$\le \|F(v_1(s),v_1(s)-v_2(s))\|_{D_\infty^{-\beta}} + \|F(v_1(s)-v_2(s),v_2(s))\|_{D_\infty^{-\beta}}$$
$$\le C_1\{\|v_1(s)\|_{D_1^{\alpha}} + \|v_2(s)\|_{D_1^{\alpha}}\}\|v_1(s)-v_2(s)\|_{D_1^{\alpha}}.$$

Hence, when v and w belong to M, by Lemma 3.2 we have

$$t^{\alpha-\gamma-\eta}\|\Phi v(t) - \Phi w(t)\|_{D_1^{\alpha}} \le C_2\|F(u_0+v,u_0+v)-F(u_0+w,u_0+w)\|_{-\beta,\infty,2\alpha-2\gamma,t}$$
$$\le 4C_1C_2K_0\|v-w\|_{\alpha,1,\alpha-\gamma,t}.$$

Therefore, with $L := 4T_0^{\eta}C_1C_2\|u_0\|_{\alpha,1,\alpha-\gamma,T_0} < 1$, we have

$$(6.7)\qquad \|\Phi v-\Phi w\|_{\alpha,1,\alpha-\gamma,T_0} \le L\|v-w\|_{\alpha,1,\alpha-\gamma,T_0}.$$

Consequently by the fixed point theorem we obtain a solution of (III).

Next, let $u \in C_{\alpha-\gamma}((0,T_0];D_1^{\alpha}(A_r))$ be a solution of (III). Then, noting that $0 < \gamma+\beta < 1$, by Lemma 3.2 and Lemma 5.2 we have

$$\int_0^t T(t-s)F(u(s),u(s))ds \in C((0,T_0];D_1^{\gamma}(A_r)),$$

$$t^{-\eta}\left\|\int_0^t T(t-s)F(u(s),u(s))ds\right\|_{D_1^{\gamma}} \le C_1\|F(u,u)\|_{-\beta,\infty,2\alpha-2\gamma,t}$$

$$\leq C_1C_2\|u\|^2_{\alpha,1,\alpha-\gamma,t} \to 0 \text{ as } t \to +0.$$

Therefore, $u \in C([0,T_0];D^{\gamma}_{\infty-}(A_r))$.

Finally, we discuss the uniqueness. Let u be a solution of (III) such that $u \in C([0,T_0];D^{\gamma}_{\infty-}(A_r)) \cap C_{\sigma-\gamma}((0,T_0];D^{\sigma}_{\infty}(A_r))$ with $\sigma > \gamma^+$. Since we can choose α sufficently near γ if $\gamma \geq 0$ and we may take $\alpha = 0$ if $\gamma < 0$, without loss of generality, we may assume that $\gamma < \alpha < \sigma$. By the interpolation inequality we have

$$\|u(t)\|_{D^{\alpha}_1} \leq C\|u(t)\|^{\theta}_{D^{\gamma}_{\infty}} \cdot \|u(t)\|^{1-\theta}_{D^{\sigma}_{\infty}} \quad \text{with } \theta = \frac{\sigma-\alpha}{\sigma-\gamma},$$

which implies that $u \in C_{\alpha-\gamma}((0,T_0];D^{\alpha}_1(A_r))$. Now the uniqueness follows from (6.7). This completes the proof of Theorem A.

Remark 6. From the above proof we see that any solution of (III) in $C_{\alpha-\gamma}((0,T_0];D^{\alpha}_1(A_r))$ for some non-negative number α with $\gamma < \alpha < \min\{1-\delta, \frac{1}{2} + \gamma, \frac{1}{2} + \frac{\gamma}{2}\}$ is unique, and belongs to $C([0,T_0];D^{\gamma}_{\infty-}(A_r))$.

7. Proof of Theorem B

The heart of the proof of Theorem B is the following lemma:

Lemma 7. Assume that $a \in D^{\gamma}_{\infty-}(A_r)$, $P_rf \in C_{1-\gamma-\delta}((0,T];D^{-\delta}_{\infty}(A_r))$, $0 < \gamma+\delta < 1$, $\delta < 1$, α and β satisfy the condition

(7.1) $\qquad \alpha \geq 0,\ \beta \geq 0,\ 2\alpha+\beta \geq \frac{n}{2r} + \frac{1}{2},\ \frac{1}{2} \leq \alpha+\beta < 1,$

and put $\mu := 1 - \max\{\beta,\delta\}$. If $u \in C((0,T];D^{\alpha}_1(A_r))$ is a solution of (III), then $u \in C^{\mu-\alpha'}((0,T];D^{\alpha'}_1(A_r))$ for any α' with $0 \leq \alpha' < \mu$.

Proof. A simple calculation shows that for any $0 < \varepsilon < T$

(7.2) $\quad u(t) = T(t-\varepsilon)u(\varepsilon) + \int_{\varepsilon}^{t} T(t-s)\{F(u(s),u(s))+P_rf(s)\}ds.$

It follows from Lemma 3.1 that $T(t-\varepsilon)u(\varepsilon) \in C^{\infty}((\varepsilon,T];D^{\alpha'}_1(A_r))$, and it follows from Lemma 3.2 that for any $0 \leq \alpha' < 1-\delta$

$$\int_{\varepsilon}^{t} T(t-s)P_rf(s)ds \in C^{1-\alpha'-\delta}((\varepsilon,T];D^{\alpha'}_1(A_r)),$$

since $P_r f \in C([\varepsilon,T];D_\infty^{-\delta}(A_r))$.

Next, since $u \in C([\varepsilon,T];D_1^\alpha(A_r))$ and α and β satisfy (7.1), by Lemma 5.2 we have $F(u,u) \in C([\varepsilon,T];D_\infty^{-\beta}(A_r))$. Hence, by Lemma 3.2 we have

$$\int_\varepsilon^t T(t-s)F(u(s),u(s))ds \in C^{1-\alpha'-\beta}((\varepsilon,T];D_1^{\alpha'}(A_r)) \text{ for any } 0 \le \alpha' < 1-\beta.$$

Since ε is arbitrary, we have the conclusion of the lemma.

We now show that straight applications of the lemma give the proof of Theorem B. Let γ, δ, a and $P_r f$ be as in Theorem A, and let u $\in C((0,T];D_1^\sigma(A_r))$ with $\sigma \ge 0$, $\sigma > \gamma$.

By π we denote the set of all pairs (α,β) satisfying (7.1). When $(\sigma,\delta) \in \pi$, by Lemma 7 we see that $u \in C^{1-\alpha-\delta}((0,T];D_1^\alpha(A_r))$ for any $0 \le \alpha < 1-\delta$. Otherwise, there is a finte sequence of numbers such that

$$\alpha_1 \le \sigma,\ (\alpha_1,\beta_1) \in \pi,\ \alpha_1 < \alpha_2 < \mu_1 := 1 - \max\{\beta_1,\delta\},$$
$$(\alpha_2,\beta_2) \in \pi,\ \beta_1 > \beta_2,\ \alpha_2 < \alpha_3 < 1 - \max\{\beta_2,\delta\},\ \cdots,$$
$$\alpha_{k-1} < \alpha_k < 1 - \max\{\beta_{k-1},\delta\},\ (\alpha_k,\delta) \in \pi.$$

Since $(\alpha_1,\beta_1) \in \pi$, $\alpha_2 < \mu_1$ and $u \in C((0,T];D_1^{\alpha_1}(A_r))$, by Lemma 7 we have $u \in C^{\mu_1-\alpha_2}((0,T];D_1^{\alpha_2}(A_r))$. Hence, considering that $(\alpha_2,\beta_2) \in \pi$ and $\alpha_3 < \mu_2$, we have $u \in C^{\mu_2-\alpha_3}((0,T];D_1^{\alpha_3}(A_r))$ by Lemma 7. Repeating this argument, we finally have $u \in C^{\mu-\alpha}((0,T];D_1^\alpha(A_r))$ for any $0 \le \alpha < \mu := 1-\delta$, and we have proved Part (i).

Proof of Part (ii). Assume now that $P_r f \in C^\nu((0,T];X_r)$, $\nu > 0$. Since $u \in C^{1-\alpha}((0,T];D_1^\alpha(A_r))$ for any $0 \le \alpha < 1$ by Part (i), and since we can choose a positive number α so that $\max\{1/2, n/4r+1/4, \gamma\} < \alpha < 1$, it follows from Lemma 5.2 that $F(u,u) \in C^{1-\alpha}((0,T];X_r)$. By (7.2), Lemma 3.2 and Remark 3 we have the conclusion of Part (ii).

8. Proof of Theorem C

For simplicity, we assume that $P_r f = 0$. The proof when $P_r f \neq 0$ is essentially the same. Let $u(t)$ be a solution of (Ⅲ) such that $u \in C((0,T];D_1^{\sigma}(A_r))$ for some non-negative number σ with $\sigma > \frac{n}{2r} - \frac{1}{2}$. Theorem B gives that $u \in C^{1-\alpha}((0,T];D_1^{\alpha}(A_r))$ for any $0 \le \alpha < 1$. Since $\frac{n}{2r} - \frac{1}{2} \le \gamma < 1$, we can choose positive numbers s and α so that $n < s < \infty$ and $0 \le \frac{n}{2r} - \frac{n}{2s} < \alpha < 1$. Hence it follows from Lemma 4.6 that

$$C^{1-\alpha}((0,T];D_1^{\alpha}(A_r)) \subset C^{1-\alpha}((0,T];D_1^{\alpha'}(A_s)) \text{ with } \alpha' = \alpha - \frac{n}{2r} + \frac{n}{2s},$$

and $\alpha' > \frac{n}{2s} - \frac{1}{2}$. By using Theorem B once more we have $u \in C^{1-\alpha}((0,T];D_1^{\alpha}(A_s))$ for any $0 \le \alpha < 1$. Thus, by replacing s by r we may assume that $r > n$ and $a \in D_{\infty-}^{\gamma}(A_r)$,

$$u \in C^{1-\alpha}((0,T];D_1^{\alpha}(A_r)) \text{ for any } 0 \le \alpha < 1, \tag{8.1}$$

and u satisfies

$$u(t) = T(t)a + \int_0^t T(t-s)F(u(s),u(s))ds. \tag{8.2}$$

Now we are going to prove the theorem. It is obvious that $T(t)a \in C^{\infty}((0,T];D_1^{\alpha}(A_r))$. As $\frac{n}{2r} + \frac{1}{2} < 1$, we can take α so that $\alpha \ge \frac{n}{2r} + \frac{1}{2}$. Then by (8.1) and Lemma 5.2 we have $F(u,u) \in C^{1-\alpha}((0,T];X_r)$. Therefore, for any $0 < \varepsilon < T$, in view of (7.2), by Lemma 3.2 we have $u \in C^{2-2\alpha}((\varepsilon,T];D_1^{\alpha}(A_r))$. As ε can be taken arbitrarily small, we have $C^{2-2\alpha}((0,T];D_1^{\alpha}(A_r))$. By repeating the above argument k times, we have

$$F(u,u) \in C^{k-k\alpha}((0,T];X_r) \text{ and } u \in C^{k+1-(k+1)\alpha}((0,T];D_1^{\alpha}(A_r)).$$

Hence, we have

$$u \in C^{\infty}((0,T];D_1^{\alpha}(A_r)), \tag{8.3}$$

$$F(u,u) \in C^{\infty}((0,T];X_r). \tag{8.4}$$

Since $\alpha > \frac{n}{2r}$, it follows from lemma 4.3 and Lemma 1 (iii) that the map: $\{u,v\} \to (u,\nabla)v$ is continuous from $D_1^{\alpha}(A_r)\times D_1^{\alpha}(A_r)$ into

$\mathbb{B}^{2\alpha-1}_{r,1}(\Omega)$. Hence, by (8.3) and Leibniz's rule we have

(8.5) $$(u(t),\nabla)u(t) \in C^{\infty}((0,T];\mathbb{B}^{2\alpha-1}_{r,1}(\Omega)),$$

which means, with the aid of Lemma 4.1, that

(8.6) $$F(u,u) \in C^{\infty}((0,T];\mathbb{B}^{2\alpha-1}_{r,1}(\Omega)).$$

Since $u_t \in C^{\infty}((0,T];\mathbb{B}^{2\alpha}_{r,1}(\Omega))$ by lemma 4.3 and (8.3), and since $A_r u(t) = F(u(t),u(t)) - u_t(t)$ (see Theorem B), Lemma 4.2 gives that $u \in C^{\infty}((0,T];\mathbb{B}^{2\alpha+1}_{r,1}(\Omega))$. By the same reasoning as in the proof of (8.5) we have $(u(t),\nabla)u(t) \in C^{\infty}((0,T];\mathbb{B}^{2\alpha}_{r,1}(\Omega))$, so we have $A_r u = F(u,u) - u_t \in C^{\infty}((0,T];\mathbb{B}^{2\alpha}_{r,1}(\Omega))$, hence $u \in C^{\infty}((0,T];\mathbb{B}^{2\alpha+2}_{r,1}(\Omega))$. Repetition of this argument finally gives that $u \in C^{\infty}((0,T];\mathbb{B}_{r,1}^{\infty}(\Omega))$, and Theorem C is proved.

References

1. Agmon, S., Douglis, A. and Nirenberg, L., Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I', Comm. Pure Appl. Math., 12, 623-727 (1959). II. ibid, 17, 35-92 (1964).
2. Borchers, W. and Miyakawa, T., L_2 decay for the Navier-Stokes flow in half-spaces', Math. Ann., 282, 139-155 (1988).
3. Borchers, W. and Sohr, H., On the semigroup of the Stokes operator for exterior domains in L_p spaces', Math. Z., 196, 415-425 (1987).
4. Fujita, H. and Kato, T., On the Navier-Stokes initial value problem 1', Arch. Rational Meth. Anal., 16, 269-315 (1964).
5. Fujiwara, D. and Morimoto, H., An L_r-theorem of the Helmholtz decomposition of vector fields', J. Fac. Sci. Univ. Tokyo, Sec., 1, 24, 685-700 (1977).
6. Giga, Y., Analyticity of the semigroup generated by the Stokes

operator in L_r spaces', Math. Z., 178, 297-329 (1981).

7. Giga, Y., Domains of fractional pawers of the stokes operator in L_r spaces', Arch. Rational Meth. Anal., 89, 251-265 (1985).
8. Giga, Y. and Sohr, H., On the Stokes operator in exterior domains', J. Fac. Sci. Univ. Tokyo, Sect. IA Math., 36, 103-130 (1989).
9. Giga, Y. and Miyakawa, T., Solution in L_r of the Navier-Stokes initial value problem', Arch. Rational Mech. Anal., 89, 267-281 (1985).
10. Heywood, J. G., The Navier-Stokes equations: On the existence, regularity and decay of solutions', Indiana Univ. Math. J., 29, 639-682 (1980).
11. Kaniel, S. and Shinbort, M., Smoothness of weak solutions of the Navier-Stokes equations', Arch. Rat. Mech. Anal., 24, 302-324 (1967).
12. Kato, T. and Fujita, H., On the nonstationary Navier-Stokes system', Rend. Sem. Mat. Univ. Padova., 32, 243-260 (1962).
13. Komatsu, H., Fractional powers of operators', Pacific J. Math., 19, 285-346 (1966).
14. Komatsu, H., Fractional powers of operators, Ⅱ, Interporation spaces', Pacific J. Math., 21, 89-111 (1967).
15. Komatsu, H., Fractional powers of operatars, Ⅲ, Negative powers', J. Math. Soc. Japan., 21, 205-220 (1969).
16. Miyakawa, T., On nonstationary solutions of the Navier-Stokes equations in an exterior domain', Hiroshima Math. J., 12, 115-140 (1982).
17. Muramatu, T., On Besov spaces of functions defined in general

regions', Publ. RIMS, Kyoto Univ., 6, 515-543 (1970).

18. Muramatu, T., On imbedding theorems for Besov spaces of functions defineded in general regions', Publ. RIMS, Kyoto Univ., 7, 261-285 (1971).
19. Muramatu, T., On Besov spaces and Sobolev spaces of generalized functions defined on a general region', Publ. RIMS, Kyoto Univ., 9, 325-396 (1974).
20. Muramatu, T., Abstract Besov Spaces relative to Non-negative Operators'. in preparation.
21. Sobolevskii, P. E., On non-stationaly equations of hydrodynamics for viscous fluid', Dokl. Akad. Nauk SSSR., 128, 45-48 (1959). (Russian)
22. Solonnikov, V. A., General boundary value problems for Douglis-Nirenberg elliptic systems which are elliptic in the sense of Douglis-Nirenberg', I. Izv. Akad. Nauk SSSR Ser. Mat., 28, 665-706 (1964), (Russian. = AMS Trasl. (2) 56, 193-232 (1966)). II. Proc. Steklov Inst. Math., 92, 233-297. (1966). (Russian)
23. Solonnikov, V. A., Estimates for solutions of nonstationary Navier-Stokes equations', J. Soviet Math., 8, 467-529 (1977).
24. Yoshida, K., Functional analysis, Springer, Berlin Heidelberg New York, 1965.